Minimum Cost Flow

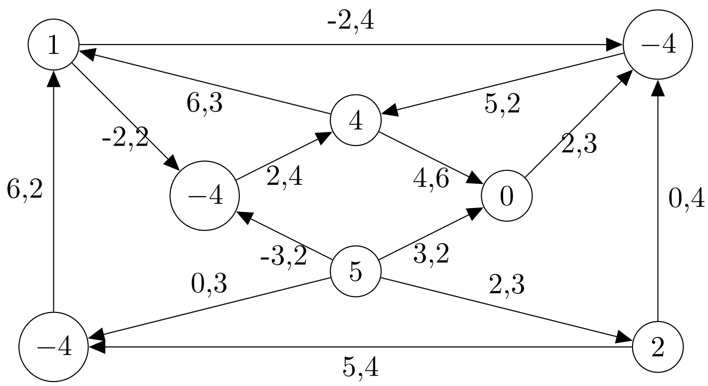
Problem: There are n coal mines and m power plants. Power plants have demands, coal mines supply coal. How to transport the coal in order to satisfy the demands and minimize the cost of transportation?

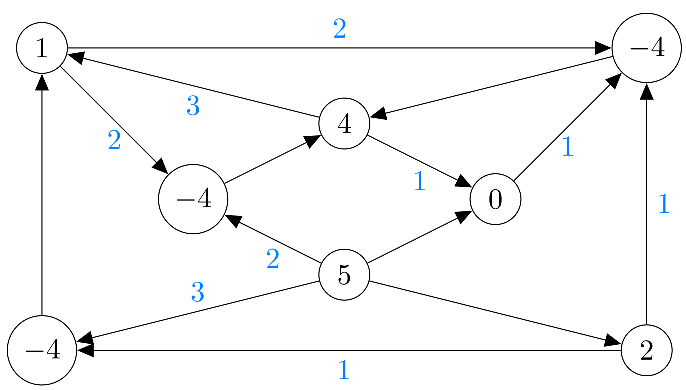
Let $G = (V, E)$ be a directed graph, $u : E \to \mathbb{R}^+$ be capacities on edges and $c : E \to \mathbb{R}$ be costs for every edge.

We call $b : V \to \mathbb{R}$ with $\sum_v b(v) = 0$ a *supply demand function* or a *boundary function*.

b-flow is a function $f : E \to \mathbb{R}^+$ such that $f(e) \le u(e)$ and $\sum_{e\in\delta^+(v)} f(e) - \sum_{e\in\delta^-(v)} f(e) = b(v)$.

1: Find a b-flow which that minimizes $\sum_e c(e)f(e))$ in the following network, where b is in every vertex, edges are labeled by c, u.





Solution: The value of the flow is 15.

If $b(v) > 0$, then b represents a *supply* at v, if $b(v) < 0$, then b represents a *demand* at v. Like flows but multiple sources and sinks.

Minimum Cost Flow Problem: find a b-flow f that minimizes $c(f) = \sum_e c(e)f(e)$.

2: Show that a b-flow f exists if and only if

$$\sum_{e\in\delta^+(X)} u(e) \ge \sum_{v\in X} b(v) \text{ for all } X \subseteq V(G). \tag{1}$$

(That is, there is always enough capacity to take excessive flow out of X.)

Solution: $\Rightarrow$ If there is a flow, the condition is easily satisfied, since $f(e) \le u(e)$.

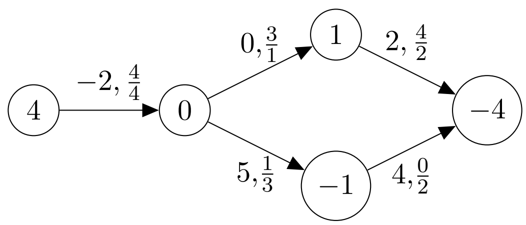
$\Leftarrow$ Suppose there is no b-flow f. Add new vertices s and t, where sv is an edge for every $v \in V$ with $b(v) > 0$. Assign $u(sv) = b(v)$. Likewise, add edge vt with $u(vt) = -b(v)$ wherever $b(v) < 0$. Call the new network (G', u', s, t).

No b-flow in G implies max-flow f' in (G', u', s, t) has value strictly less than $(\sum_v |b(v)|)/2$. The max-flow f' gives a min-cut X. The cut X violates (1).

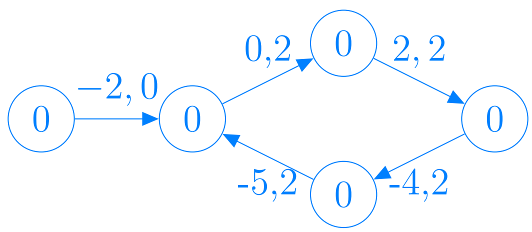
Consequence: It is possible to detect when there is no solution (no b-flow exists).

A *circulation* is a flow in a network where $b(v) = 0$ for every vertex.

3: Let f and f' be two b-flows. Consider their *difference* $f - f'$ and show that it is a circulation. Try on example first: Edge labels are $c, \frac{f}{f'}$. Compute $c(f)$, $c(f')$, find the difference.



Solution: $c(f) = 5$, $c(f') = 19$. Cost difference $c(f) - c(f') = -14$. In picture, we assign the $f - f'$ to each edge, if the difference is negative, flip the edge, negate the cost ,and assign $f' - f$. The picture gives a circulation with cost -14.



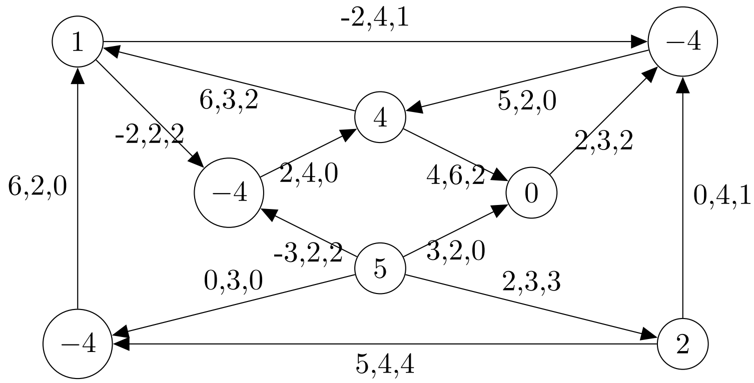
In general, the difference between two flows f and f' is a circulation whose cost is the difference in the costs of f and f'. Since f and f' are b-flows, we get $\sum_{e \in \delta^+(v)}(f(e) - f'(e)) - \sum_{e \in \delta^-(v)}(f(e) - f'(e)) = b(v) - b(v) = 0$, Hence the difference is a circulation. Summing the difference gives the difference in costs.

Algorithm: Minimum Cost Flow

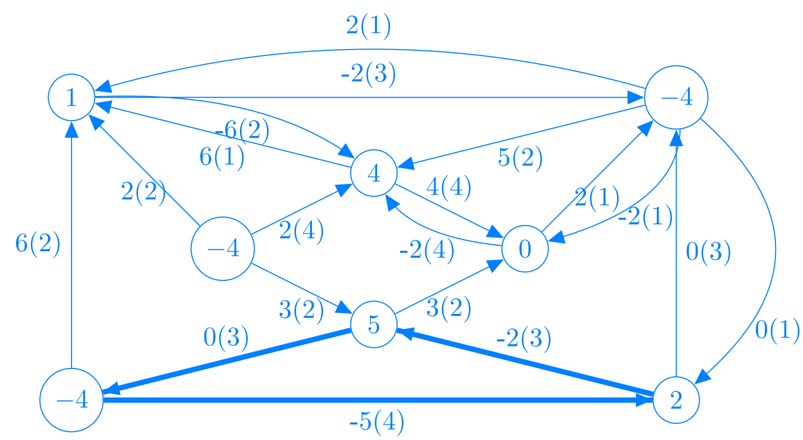
1. let f be any b-flow
2. while there exists a negative cost cycle C in the residual graph G_f,
3. pick C of minimum mean cost $= \frac{\sum_{e \in C} c(e)}{|C|}$.
4. augment on C

Minimum mean cost cycle gives polynomial time $O(m^2 n^2 \log n)$ (picking any cycle - same problem as Ford-Fulkerson).

4: Find a negative cycle in a residual graph for the following, where on edge is c, u, f in this order.



Solution: Residual graph is created by keeping edges e with nonzero $u(e) - f(e)$ and adding reverse edges if $f(e) > 0$. In order to find a negative cycle, we only need to know the cost. It is depicted in in bold.



5: Show that the algorithm is correct when it finishes. That is, f is an optimal b-flow iff it has no negative cycle.

Solution: $\Rightarrow$ If there is a negative cycle, augmenting on it, it will decrease the cost of f, contradiction with the optimality of f.

$\Leftarrow$ If there is a better flow f', the difference between flows is a circulation. Since the sum of the circulation is negative, it must contain a negative circuit.

6: How can we find a minimum mean cycle?

Solution: Wait for the next worksheet.